Problem 1.5.2

Let A be a minimal set in a dynamical system (X, f). Show that $\overline{f[A]} = A$ and that $f[A] = A$ if A is compact.

Proof of first part

Proof. Since $f[A] \subseteq A$ then $\overline{f[A]} \subseteq \overline{A}$. By the minimality of A, $\overline{A} = A$. Then, $\overline{f[A]} \subseteq A$.

By the invariance of A, $f[A]$ is invariant [Prop 1.2.1 (2)]. By the invariance of $f[A]$, $\overline{f[A]}$ is invariant [Prop 1.2.3]. Since $\overline{f[A]} \subseteq A$, $\overline{f[A]} = \varnothing$ or $\overline{f[A]} = A$. By the minimality of A, $A \neq \varnothing$. Thus the image of A is non-empty. Since $f[A] \subseteq \overline{f[A]}$, $\overline{f[A]} = A$. □

Proof of second part

Proof. Assume A is compact. By the continuity of f, $f[A]$ is compact. Then, $f[A] = \overline{f[A]}$. So, by the previous part, $f[A] = A$. □

Proposition 1.2.1

(2)

For every $k \in \mathbb{N}$ the set $f^k[A]$ is invariant. Similarly, if A is completely invariant then so is $f^k[A]$.

Proposition 1.2.3

The closure of an invariant set is invariant. If a set is completely invariant and its closure is compact, or f is a homeomophism, then the closure is completely invariant as well.

Miscellaneous Proofs

Containment and Closures

$A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$.

Proof. $\overline{B}$ is the smallest closed set that contains B. $A \subseteq B \subseteq \overline{B}$. $\overline{A}$ is the smallest closed set that contains A. Since $\overline{B}$ is a closed set that contains A, $\overline{A} \subseteq \overline{B}$.